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COCORAM

Livrable R3 : Note sur la méthodologie de synthèse

Cette note détaille la méthodologie de synthèse développée pour la conception conjointe des éléments du système

Partenaires du projet :

Note sur la méthodlogie de synthèse

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1 Matching problem

The problem of matching is considered as minimizing the reflection of the power that is intended to be transmitted to a given load (L) within a given frequency band. The load is represented as a 2-port device in Figure 1. Usually this power is transmitted to the load through a filter that avoid clutter signals to be transmitted. Both devices, the filter together with the load compose the global system (G) represented in Figure 1. Considering the matching problem, it is possible to maximize the power transmitted to the load by a smart design of the filter (F).

Throughout this work, we will consider ω as the frequency variable instead of $s = j\omega$ and the frequency axis the real axis of ω. Then the roots of the denominator of each function (e, q, ψ) lies in the upper half plane ($\mathrm{Im}(\omega) > 0$). Consequently the definition of $p^*(\omega)$ for every polynomial will be :

$$p^*(\omega) = \overline{p(\overline{\omega})} \tag{1}$$

What means that $p^*(\omega)$ is a polynomial that coincide with $\overline{p(\omega)}$ at ω real.

We will also represent the Belevitch form [Belevitch, 1968] of each system as follows (using the last definition for n^*, p^* y φ^* and omitting the ω variable to economy the notation):

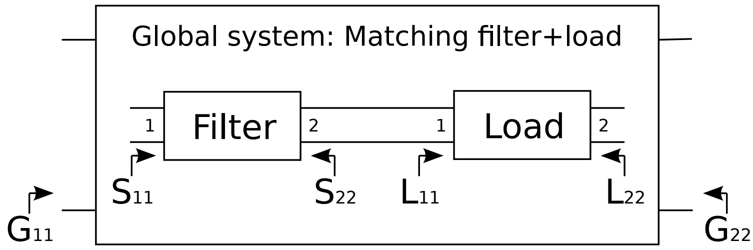


Figure 1: Global system composed of matching filter and load

$$G = \frac{1}{e}\begin{pmatrix} -\varepsilon n^* & \varepsilon m^* \\ m & n \end{pmatrix} \qquad S = \frac{1}{q}\begin{pmatrix} -\varepsilon p^* & \varepsilon r^* \\ r & p \end{pmatrix} \qquad L = \frac{1}{\psi}\begin{pmatrix} -\varepsilon \varphi^* & \varepsilon \rho^* \\ \rho & \varphi \end{pmatrix} \tag{2}$$

with ε a complex constant ($\varepsilon \in \mathbb{C}$). Here in after we will assume $\varepsilon = 1$. Moreover according to the Feldtkeller equation [Carlin and Civalleri, 1997]:

$$ee^* = nn^* + mm^* \qquad qq^* = pp^* + rr^* \qquad \psi\psi^* = \varphi\varphi^* + \rho\rho^* \tag{3}$$

The aim of matching is to minimize the reflection at the input of the global system G_{11} (Figure 1). The reflection coefficient of the global system G_{11} is obtained as [Baratchart et al., 2014]:

$$G_{11} = S_{11} + \frac{S_{21}S_{12}L_{11}}{1 - S_{22}L_{11}} = det(S)\frac{S_{22}^* - L_{11}}{1 - S_{22}L_{11}} \tag{4}$$

It means that if $S_{22}^* = L_{11}$ the reflected waves in the port 1 of the matching filter are canceled and then the reflection coefficient of the global system vanishes.

We can also write the polynomial n, m and e of the global system in term of the polynomial p, r and q of F and φ, ρ and e of L:

$$n = q\varphi + p\psi^* \tag{5}$$

$$e = q\psi + p\varphi^* \tag{6}$$

$$m = r\rho \tag{7}$$

Let define a matching point ω_n as a point where the global system G has a reflection zero ($n(\omega_n) = 0$):

$$G_{11}(\omega_{n_k}) = G_{22}(\omega_{n_k}) = 0 \longleftrightarrow S_{22}(\omega_{n_k}) = L_{11}^*(\omega_{n_k}) \quad k \in [1, N] \tag{8}$$

In the same way, minimize the reflection at the input of the global system $|G_{11}|^2$ will be totally equivalent as minimizing the pseudo-hyperbolic distance (D) between S_{22} and L_{22}^*.

$$D(h, v) = \left| \frac{h - v}{1 - hv^*} \right| \tag{9}$$

Then:

$$\min_{p,q} |G_{11}(\omega)| = \min_{p,q} \left| \frac{S_{22}(\omega) - L_{11}^*(\omega)}{1 - S_{22}(\omega)L_{11}(\omega)} \right| \tag{10}$$

Therefore, the matching problem can be faced from two different points of view: by minimizing the pseudohyperbolic distance between S_{22} and L_{22}^* or by designing the global filter G.

Problem 1 *Minimizing the pseudohyperbolic distance between S_{22} and L_{22}^*.*

Given the transmission polynomial r of the filter (F), find:

$$\min_{p,q} \max_{\omega} \left| \frac{\frac{p}{q}(\omega) - L_{11}^*(\omega)}{1 - \frac{p}{q}(\omega)L_{11}(\omega)} \right| \qquad \omega \in I \tag{11}$$

Subject to:

$$qq^* = pp^* + rr^* \tag{12}$$

considering that I is the frequency interval where the filter should match the load.

The second point of view consist in the design of the system G as a filter with the desired specification. Next the load L is dechained from the global system to retrieve the matching filter as follows:

$$p = \frac{n\psi - e\varphi}{|\rho|^2} \tag{13}$$

$$q = \frac{e\psi^* - n\varphi^*}{|\rho|^2} \tag{14}$$

$$r = m/\rho \tag{15}$$

This approach is the same as the one proposed in [Lefteriu et al., 2013] and in order to be able to extract p, q, and r it is important to satisfy the conditions (16) and (17).

$$G_{22}(\omega_{\rho_i}) = L_{22}(\omega_{\rho_i}) \tag{16}$$

$$\left.\frac{\partial G_{22}}{\partial \omega}(\omega)\right|_{\omega=\omega_{\rho_i}} = \left.\frac{\partial L_{22}}{\partial \omega}(\omega)\right|_{\omega=\omega_{\rho_i}} \tag{17}$$

In other words, the value and the first derivative of G_{22} must equal the value and the first derivative of L_{22} at every transmission zero of the load ω_{ρ_i} (Including those at infinity). Then it is possible to formulate the second problem:

Problem 2 *Synthesis of the global filter G*
Given the transmission polynomial m of the system (G), find:

$$\underset{n,e}{min}\ \underset{\omega}{max}\,(|G_{22}|) = \underset{n,e}{min}\ \underset{\omega}{max}\left(\left|\frac{n}{e}\right|\right) \qquad \omega \in I \tag{18}$$

Subject to:

$$ee^* = nn^* + mm^* \tag{19}$$

$$G_{22}(\omega_{\rho_i}) = L_{22}(\omega_{\rho_i}) \tag{20}$$

$$\left.\frac{\partial G_{22}}{\partial \omega}(\omega)\right|_{\omega=\omega_{\rho_i}} = \left.\frac{\partial L_{22}}{\partial \omega}(\omega)\right|_{\omega=\omega_{\rho_i}} \qquad \forall \omega_{\rho_i} : L_{21}(\omega_{\rho_i}) = 0 \tag{21}$$

where I represent the passband of the desired system G.

2 Bounds for the reflection level by solving the problem 2 (Design of the global filter G)

Solving the problem 2 corresponds to the design of the best filter G that satisfies (16) and (17) in order to be able to extract the load. This can be regarded as the approach equivalent to the one presented in [Lefteriu et al., 2013]. We solve now this problem for a load of first order with a transmission zero at infinity ($\omega_\rho = \infty$). According to the notation in Figure 1, we can consider that the polynomials n and e of the global system (G) are monic. Otherwise it is possible to normalize all polynomial (n, e and m) such as n and e become monic and the scattering parameters (G) remain the same. In the same way we consider for the load that φ and ψ are monic as well.

Consequently, at the transmission zero of the load ($\omega_{\rho_i} = \infty$), the reflection coefficient G_{22} will take the same value as the parameter L_{22} of the load. Thus (16) will be satisfied by construction. It is only necessary to satisfy (17) in order to extract the load. Therefore the problem is to find a filter of order N (where N is the number of matching point ω_{n_i}) with the minimum reflection level in the passband whose derivative met (21) at infinity:

$$\underset{n,e}{min}\ \underset{\omega}{max}|G_{22}(\omega)| \qquad \omega \in I \tag{22}$$

subject to (17). Here, in order to compute the derivative at $\omega = \infty$, we evaluate at $\omega = 0$ the derivative of $G_{22}\left(\frac{1}{\omega}\right)$ and $L_{22}\left(\frac{1}{\omega}\right)$:

$$\left.\frac{\partial G_{22}}{\partial \omega}\left(\frac{1}{\omega}\right)\right|_{\omega=0} = \left.\frac{\partial L_{22}}{\partial \omega}\left(\frac{1}{\omega}\right)\right|_{\omega=0} \tag{23}$$

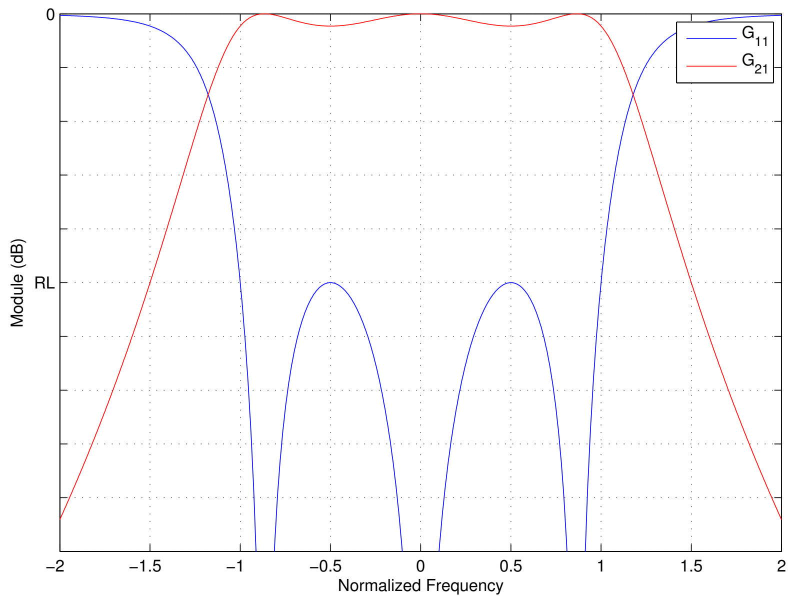


Figure 2: Lowpass prototype of a Tchebyshev filter.

With the definition:

$$n = n_N\omega^N + n_{N-1}\omega^{N-1} + n_{N-2}\omega^{N-2} + \cdots + n_1\omega + n_0 \tag{24}$$

$$e = e_N\omega^N + e_{N-1}\omega^{N-1} + e_{N-2}\omega^{N-2} + \cdots + e_1\omega + e_0 \tag{25}$$

the derivative at 0 of $G_{22}\left(\frac{1}{\omega}\right)$ is:

$$\frac{\partial}{\partial\omega}G_{22}\left(\frac{1}{\omega}\right)\Big|_{\omega=0} = \frac{n'e - e'n}{e^2}\left(\frac{-1}{\omega^2}\right)\Big|_{\omega=0} = \frac{e_N n_{N-1} - n_N e_{N-1}}{e_N^2} \tag{26}$$

and since both n and e are monic this derivative is equal to the sum of the poles minus the sum of the zeros of G_{22}.

$$\frac{\partial}{\partial\omega}G_{22}\left(\frac{1}{\omega}\right)\Big|_{\omega=0} = n_{N-1} - e_{N-1} = \sum_{i=1}^{N}\omega_{e_i} - \sum_{i=1}^{N}\omega_{n_i} \ : e(\omega_{e_i}) = 0,\ n(\omega_{n_i}) = 0 \tag{27}$$

We regard now the Feldtkeller equation (3) with the assumptions that the reflection polynomial n and the denominator (e) are monic ($e_N = n_N = 1$) and that the transmission polynomial (m) has at least one transmission zero at infinity ($m_N = m_{N-1} = 0$). Lets define the polynomial $U = ee^* = U_{2N}\omega^{2N} + \cdots + U_0$. Then:

$$U_{2N} = e_N e_N^* = 1 \tag{28}$$

$$U_{2N-1} = 2 \cdot Re\left(e_N e_{N-1}^*\right) = 2 \cdot Re\left(n_N n_{N-1}^*\right) \longrightarrow Re\left(e_{N-1}\right) = Re\left(n_{N-1}\right) \tag{29}$$

Thus the value $n_{N-1} - e_{N-1}$ will be pure imaginary. Similarly for the load we will obtain a pure imaginary derivative (jh) (where h is real):

$$\frac{\partial}{\partial\omega}L_{22}\left(\frac{1}{\omega}\right)\Big|_{\omega=0} = \frac{\varphi'\psi - \psi'\varphi}{\psi^2}\Big|_{\omega=0} = \frac{\psi_N\varphi_{N-1} - \varphi_N\psi_{N-1}}{\psi_N^2} = jh \tag{30}$$

2.1 Global system as a Tchebyshev filter

If the global filter G is Tchebyshev (Figure 2), then there will be N matching points in the real axis. Although it will be shown that this filter is not the best option for matching, it achieves the best rejection in the stopband given a level the reflection level in the band.

According to (27) the derivative at infinity can be computed as the difference between the sum of the roots of e and the sum of the roots of n. For a Tchebyshev filter, n is symmetrical and real, thus the sum of its roots will be zero. Meanwhile the roots of e are the zeros of the denominator of:

$$|G_{22}|^2 = \frac{1}{1 + \frac{\epsilon^2}{T_N^2}} \tag{31}$$

Where $T_N(\omega) = cos(N \cdot arccos(\omega))$ is the Tchebyshev polynomial of degree N and ϵ is equal to the transmission constant (m) times the leading coefficient of T_N. In this way n remains monic. The roots of the denominator are computed as following [Carlin and Civalleri, 1997]:

$$1 + \frac{\epsilon^2}{T_N^2} = 0 \longrightarrow T_N = cos(N \cdot arccos(\omega_{z_k})) = \pm j\epsilon \tag{32}$$

$$\begin{aligned} \omega_{z_k} &= cos\left(\frac{1}{N} arccos(\pm j\epsilon)\right) = cos\left(\frac{1}{N}\left(arcsin(\pm j\epsilon) - \frac{\pi}{2}\right)\right) = \\ &= cos\left(\frac{\mp j}{N}\left(arcsinh(\epsilon) - j\frac{\pi}{2}\right)\right) = cosh\left(\frac{1}{N}\left(ln\left(\epsilon + \sqrt{\epsilon^2+1}\right) + jk\pi - j\frac{\pi}{2}\right)\right) \end{aligned} \tag{33}$$

defining $\theta_k = \frac{\pi}{2N}(2k-1)$ and $X = \epsilon + \sqrt{\epsilon^2+1}$:

$$\omega_{z_k} = \frac{1}{2}\left(X^{1/N} e^{j\theta_k} + X^{-1/N} e^{-j\theta_k}\right) \tag{34}$$

As it is already commented, the sum of the roots will be pure imaginary and so will the derivative jh:

$$jh = \sum_{k=1}^{N} \omega_{z_k} = \frac{1}{2}\left(X^{1/N} \sum_{k=1}^{N} e^{j\theta_k} + X^{-1/N} \sum_{k=1}^{N} e^{-j\theta_k}\right) \tag{35}$$

Expanding both summations we get:

$$\sum_{k=1}^{N} e^{j\theta_k} = \frac{j}{sin\left(\frac{\pi}{2N}\right)} \qquad \sum_{k=1}^{N} e^{-j\theta_k} = \frac{-j}{sin\left(\frac{\pi}{2N}\right)} \tag{36}$$

Then

$$jh = \sum_{k=1}^{N} \omega_{z_k} = \frac{1}{2}\left(\frac{jX^{1/N}}{sin\left(\frac{\pi}{2N}\right)} + \frac{-jX^{-1/N}}{sin\left(\frac{\pi}{2N}\right)}\right) \tag{37}$$

$$h \cdot sin\left(\frac{\pi}{2N}\right) = \frac{1}{2}\left(X^{1/N} - X^{-1/N}\right) = sinh\left(\frac{1}{N} ln(X)\right) = sinh\left(\frac{1}{N} arcsinh(\epsilon)\right) \tag{38}$$

$$\epsilon = sinh\left(N \cdot arcsinh\left(h \cdot sin\left(\frac{\pi}{2N}\right)\right)\right) \tag{39}$$

The best reflection level in this case is computed as:

$$RL(dB) = 10 \cdot log_{10}\left(\frac{1}{1+\epsilon^2}\right) \tag{40}$$

This reflection level (RL) is monotonous decreasing with ϵ. With regard to the derivative:

$$\left.\frac{\partial}{\partial\omega} G_{22}\left(\frac{1}{\omega}\right)\right|_{\omega=0} = \frac{1}{sin\left(\frac{\pi}{2N}\right)} sinh\left(\frac{1}{N} arcsinh(\epsilon)\right) \tag{41}$$

it is monotonous increasing since both functions ($sinh$ and $arcsinh$) are monotonous. In consequence, the reflection level will be better the higher the value of h is. Equivalently if we have a filter F^1 and

$$\left|F_{11}^1\right| = L_1 \tag{42}$$

$$\left.\frac{\partial}{\partial\omega} F_{22}^1\left(\frac{1}{\omega}\right)\right|_{\omega=0} < h \tag{43}$$

we can increase the value of epsilon until equation (43) becomes equality. As a result, we will obtain another filter F^2 that matches the derivative of the load (h) and provides a better reflection level (RL):

$$\left|F_{11}^2\right| = L_2 < L_1 \tag{44}$$

$$\left.\frac{\partial}{\partial\omega} F_{22}^2\left(\frac{1}{\omega}\right)\right|_{\omega=0} = h \tag{45}$$

Theorem 1 *Given a load of order 1, the best matching filter is the filter in which the sum of the roots of e minus the sum of the roots of n is the smallest possible since it will match a load that provides an smaller derivative (h):*

$$Im\left(\sum_{i=1}^{N}\omega_{e_i} - \sum_{i=1}^{N}\omega_{n_i}\right) \le h \tag{46}$$

Then if the constraint (46) is not saturated, then it is possible to increase the transmission constant m to improve the reflection level until the constraint is saturated.

Figure 3 (Lines from red to blue) shows that the reflection level (G_{22}) will be better the higher the derivative is. In fact if the derivative is infinity, then there is no constraint about the reflection level. Moreover, it can be seen that for a given derivative, there is a limit on the possible reflection level regardless of the filter order. If it is required a better level, it is necessary to design another kind of filter with complex reflection zeros. Then the matching point (ω_n's) will not be in the frequency axis but in the complex plane.

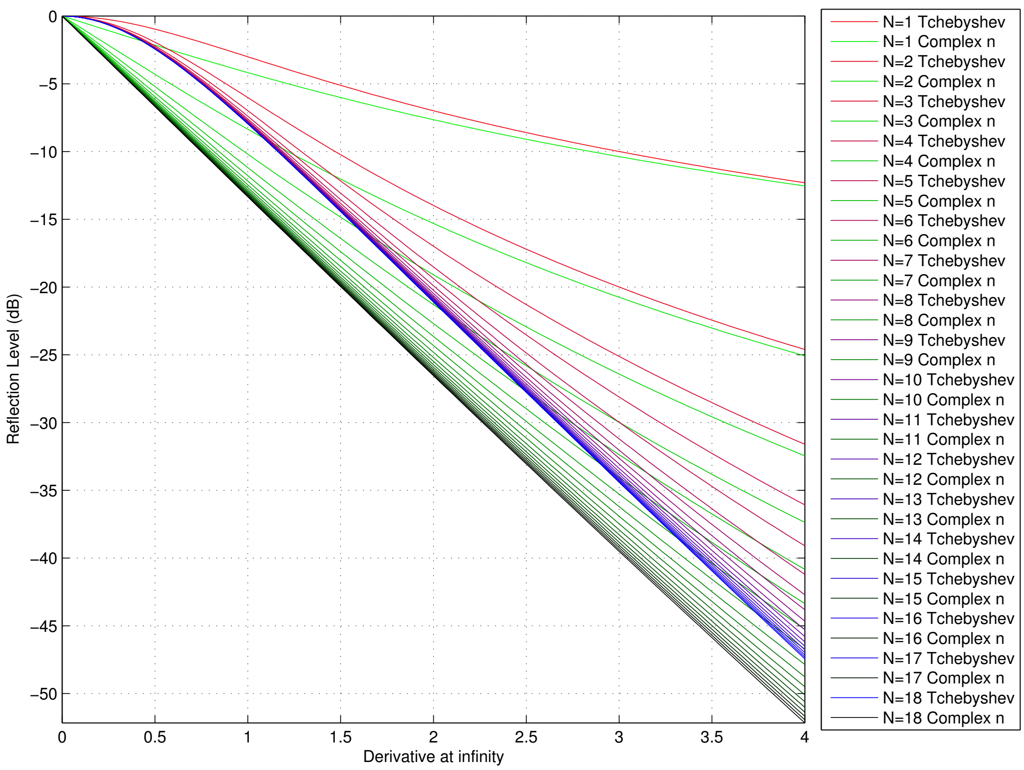


Figure 3: Best reflection level for each filter order with respect to the derivative at infinity of the antenna. (From red to blue: Tchebyshev filter; From green to black: equioscillating filter.)

2.2 Global system as an equioscillating filter

It is also possible to consider a response $G_{22} = \frac{n}{e}$ where both n and e are complex polynomials. Then we can get a better level that the one provided by the Tchebyshev filter by considering a response in which the transmission polynomial $|G_{22}|^2$ equioscillates between two levels λ_1 and λ_2 (Figure 4).

It can be proved that given the levels λ_1 and λ_2, there exist one unique filter G (in modulus) with a parameter $|G_{22}|^2$ that oscillates between λ_1 and λ_2 in the passband or equivalently with a transmission coefficient $|G_{21}|^2$ that oscillates between γ_1 and γ_2 ($\lambda_1 = 1 - \gamma_1$, $\lambda_2 = 1 - \gamma_2$).

$$|G_{22}|^2 = \left|\frac{n}{e}\right|^2 \qquad\qquad |G_{21}|^2 = \left|\frac{m}{e}\right|^2 \tag{47}$$

First, we compute a Tchebyshev filter (F) with a ripple level of γ_2/γ_1:

$$F = \frac{1}{F_e}\begin{pmatrix} -F_n^* & F_m \\ F_m & F_n \end{pmatrix} \tag{48}$$

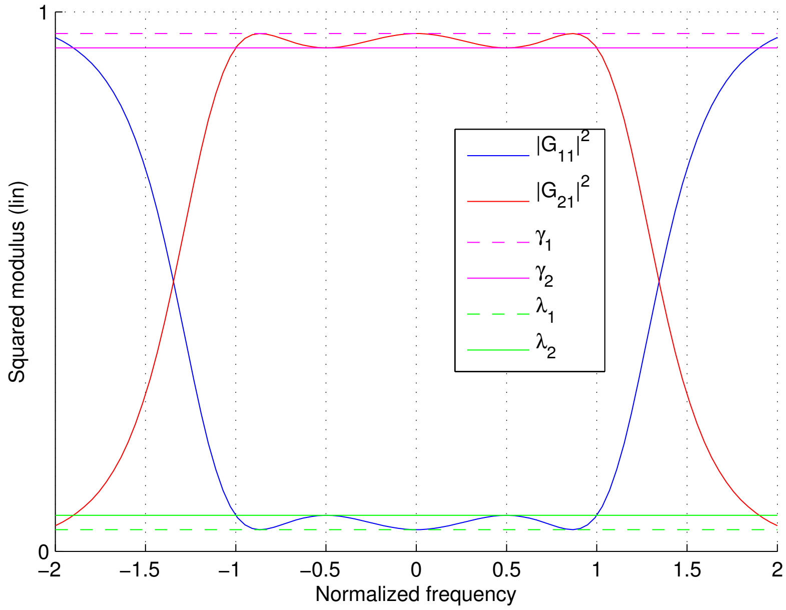


Figure 4: Lowpass prototype of an equioscillating filter.

We consider for this filter that all polynomials are normalized to get $F_m = 1$. It means that the polynomials F_n and F_e are not monic. For this filter, the parameter $|F_{21}|^2$ will oscillate between 1 and γ_2/γ_1 within the passband ($\gamma_1 > \gamma_2$). It takes the expression:

$$|F_{21}|^2 = \left|\frac{F_m}{F_e}\right|^2 = \frac{1}{1 + \left(\frac{\gamma_1}{\gamma_2} - 1\right) T_N^2} \tag{49}$$

The transmission coefficient $|G_{21}|^2$ that equioscillates between γ_1 and γ_2 is obtained as:

$$|G_{21}|^2 = \gamma_1 |F_{21}|^2 = \frac{1}{|F_e/\gamma_1|^2} = \left|\frac{m}{e}\right|^2 \tag{50}$$

Then the polynomials $|e|^2$, $|n|^2$ and $|m|^2$ take the form:

$$|m|^2 = 1 \tag{51}$$

$$|e|^2 = \frac{|e'|^2}{\gamma_1} = \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right) T_N^2 + \frac{1}{\gamma_1} \tag{52}$$

$$|n|^2 = |e|^2 - |m|^2 = \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right) T_N^2 + \left(\frac{1}{\gamma_1} - 1\right) \tag{53}$$

or equivalently (dividing everything by $\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)$):

$$|m|^2 = \frac{\gamma_1\gamma_2}{\gamma_1 - \gamma_2} \tag{54}$$

$$|n|^2 = T_N^2 + \left(\frac{\gamma_2(1-\gamma_1)}{\gamma_1 - \gamma_2}\right) = T_N^2 + \alpha^2 \tag{55}$$

$$|e|^2 = T_N^2 + \left(\frac{\gamma_2}{\gamma_1 - \gamma_2}\right) = T_N^2 + \beta^2 \tag{56}$$

This proves that the modulus of the filter G (meaning the functions $|n|^2$, $|m|^2$ and $|e|^2$) is unique given the values of γ_1 and γ_2.

Now, there exist different possibilities to obtain n and e, either with the roots in the upper half plane or in the lower half plane For e it is necessary to take the solution with the root in the upper half plane in order for the system to be stable. With regard to the roots of n, the roots will be also in the upper half plane according to Theorem 1 in order to match the smallest derivative possible.

The obtained polynomials n and e are just a shifted version of the Tchebyshev polynomial and can be obtained as a function of two constants α and β.

The objective then is to get the values of α and β that satisfy (17). The expressions (55) and (56) take the same form as the Feldtkeller equation. Thus the roots of n and e will have an expression analogous to (34).

$$T_N^2(\omega_{n_k}) = \pm j\alpha \longrightarrow \omega_{n_k} = \frac{1}{2}\left(X_\alpha^{1/N} e^{j\theta_k} + X_\alpha^{-1/N} e^{-j\theta_k}\right); \qquad X_\alpha = \alpha + \sqrt{\alpha^2+1} \tag{57}$$

$$T_N^2(\omega_{e_k}) = \pm j\beta \longrightarrow \omega_{e_k} = \frac{1}{2}\left(X_\beta^{1/N} e^{j\theta_k} + X_\beta^{-1/N} e^{-j\theta_k}\right); \qquad X_\beta = \beta + \sqrt{\beta^2+1} \tag{58}$$

In order to be able to extract the load afterward, the difference between the sum of the roots of e and the sum of the roots of p must be equal to the derivative of the load at infinity (jh):

$$\begin{aligned} jh &= \sum_{k=1}^{N} \omega_{e_k} - \sum_{k=1}^{N} \omega_{n_k} \\ &= \frac{1}{2}\left(X_\beta^{1/N} \sum_{k=1}^{N} e^{j\theta_k} + X_\beta^{-1/N} \sum_{k=1}^{N} e^{-j\theta_k}\right) - \frac{1}{2}\left(X_\alpha^{1/N} \sum_{k=1}^{N} e^{j\theta_k} + X_\alpha^{-1/N} \sum_{k=1}^{N} e^{-j\theta_k}\right) \end{aligned} \tag{59}$$

Using (36):

$$h = \frac{1}{2}\left(\frac{X_\beta^{1/N}}{sin\left(\frac{\pi}{2N}\right)} - \frac{X_\beta^{-1/N}}{sin\left(\frac{\pi}{2N}\right)}\right) - \frac{1}{2}\left(\frac{X_\alpha^{1/N}}{sin\left(\frac{\pi}{2N}\right)} - \frac{X_\alpha^{-1/N}}{sin\left(\frac{\pi}{2N}\right)}\right) \tag{60}$$

$$\begin{aligned} h \cdot sin\left(\frac{\pi}{2N}\right) &= sinh\left(\frac{1}{N} ln\left(X_\beta\right)\right) - sinh\left(\frac{1}{N} ln\left(X_\alpha\right)\right) \\ &= sinh\left(\frac{1}{N} arcsinh\left(\beta\right)\right) - sinh\left(\frac{1}{N} arcsinh\left(\alpha\right)\right) \end{aligned} \tag{61}$$

$$\beta = \sinh\left(N \cdot arcsinh\left(sinh\left(\frac{1}{N} arcsinh\left(\alpha\right)\right) + h \cdot sin\left(\frac{\pi}{2N}\right)\right)\right) \tag{62}$$

The reflection level remains as:

$$RL(dB) = 10 \cdot log_{10}\left(\left|G_{22}(\omega=1)\right|^2\right) = 10 \cdot log_{10}\left(\frac{1+\alpha^2}{1+\beta^2}\right) \tag{63}$$

Optimizing with respect to α, it is possible to get the best reflection level with this kind of response for a given order and load. Figure 3 (lines from green to black) shows the reflection level that can be obtained in this way for each filter order as a function of the derivative at infinity of the antenna (lines from green to black). It is also possible to compare this levels with the ones provided by a Tchebyshev response (lines from red to blue). We can see that the result with a complex n is always better that the level obtained with a Tchebyshev filter what proves the non-optimality of the second one. Moreover it can also be noted that when the derivative tend to infinity, the result obtained with this equioscillating filter tends to the Tchebyshev one. That was obvious since when the derivative tend to infinity, there is no constraint over n and e. Then the best filter is the Tchebyshev one.

3 Minimization of the pseudohyperbolic distance

In the previous problem, the global filter is designed first with the constrains that allows to extract afterwards the load from it. However another point of view is to design the matching filter that minimize the hyperbolic distance between S_{22} and L_{11}. Then both the filter and the load are chained to obtain the global system G.

3.1 Point-wise matching problem

If we consider a problem of point-wise matching, the objective is to build a function $S_{22} = p/q$ satisfying the Feldtkeller equation (3) that matches L_{11}^* (solve (64)) in the larger possible amount of points (ω_k):

$$\left|\frac{S_{22}(\omega_k) - L_{11}^*(\omega_k)}{1 - S_{22}(\omega_k) L_{11}(\omega_k)}\right| = 0 \tag{64}$$

We come to the following interpolation problem.

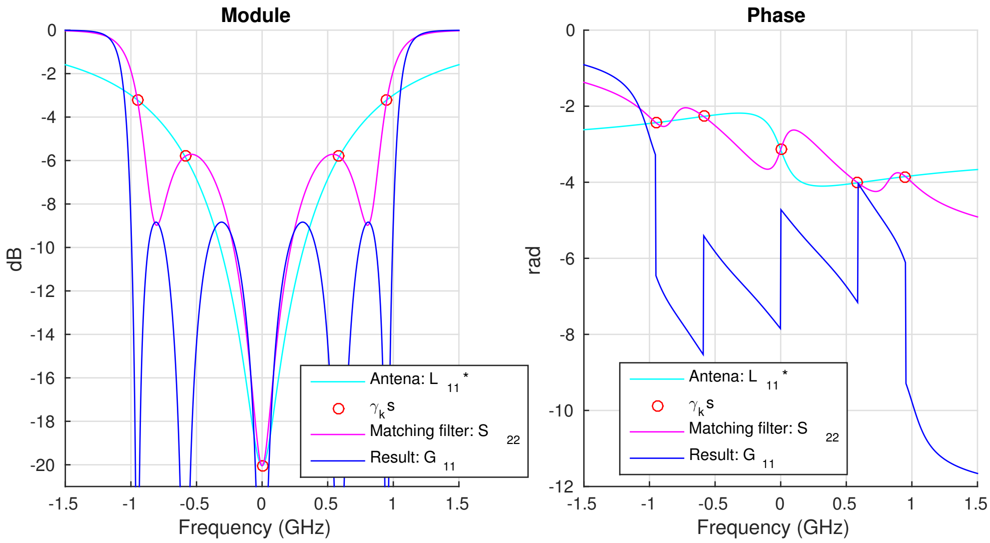


Figure 5: Result of the Point-Wise matching problem.

Problem 3 (Interpolation problem) *Given a nonzero polynomial r of degree at most N, find a polynomial of degree less that N that satisfies:*

$$\frac{p}{q}(\omega_k) = \gamma_k \qquad |\gamma_k| < 1 \qquad \omega_k \in \mathbb{R} \qquad k \in [1, N] \tag{65}$$

$$qq^* = pp^* + rr^* \qquad q \ stable \tag{66}$$

where $\gamma_k = L^*_{11}(\omega_k)$, r has no zeros at the points ω_k's and q is normalized to obtain $Re(q(\alpha)) > 0$ (α in the lower half plane). This is known to be possible in one more point than the order of p [Baratchart et al., 2014] leading to the following statement:

Theorem 2 *There exist an unique function $S_{22} = \frac{p}{q}$ stable of degree at most $N-1$ that satisfies (65) in N different real points ω_k's.*

This function S_{22} can be found by continuation, starting from a polynomial p' that provided an known output Γ'

$$\Gamma' = (\gamma'_1, \cdots, \gamma'_N) \quad \gamma'_k = \frac{p'}{q'}(\omega_{n_k}) \tag{67}$$

and following a given path between Γ' and $\Gamma = (\gamma_1, \cdots, \gamma_2)$, it is possible to obtain by prediction-correction the polynomial p that gives Γ. Figure 5 shows the result obtained when considering 5 interpolation points from a given load. Indeed the interpolation condition are satisfied with a function of degree 4 (pink line).

3.2 Minimization of the pseudohyperbolic distance

The problem with the previous approach is that the optimum matching points ω_{n_k}'s are not known a priori for an arbitrary load. To overcome that problem we can just consider Problem 1 namely minimizing the maximum level of the reflection coefficient G_{11} (Figure 1) within the frequency interval I between ω_1 and ω_2.

$$\underset{p,q}{Min}\underset{\omega}{Max} \, |S_{22}(\omega) - L^*_{11}(\omega)| \quad \forall \omega \in I \tag{68}$$

or equivalently

$$\underset{n,e}{Min}\underset{\omega}{Max} \left(|G_{11}|^2\right) = \underset{n,e}{Min}\underset{\omega}{Max} \left(\frac{1}{1 + \left|\frac{m}{n}\right|^2}\right) = \underset{n,e}{Min}\underset{\omega}{Max} \left(\left|\frac{n}{m}\right|^2\right) \tag{69}$$

Introducing the expressions for n and m from (5) and (7) the problem remains

$$Find: \ \underset{p,q}{Min}\underset{\omega}{Max} \left(\frac{|q(\omega)\varphi(\omega) + p(\omega)\psi^*(\omega)|^2}{\rho^2(\omega) r^2(\omega)}\right) \quad \forall \omega \in I \tag{70}$$

subject to

$$qq^* = pp^* + rr^* \tag{71}$$

Expression 70 will be more convenient than the one in problem 1 for the practical implementation, however a rational approximation of the load is needed in order to work with the polynomial numerator and denominator of L_{11}.

If we combine both problems, we can chose the points ω_n's to have either a Tchebyshev filter or an equioscillating response and then, use the polynomials obtained by solving Problem 2 as initial point for (70). In this way we will obtain a response, similar to the one that equioscillates, that is locally optimum.

As an example, we consider an antenna that gives a derivative at infinity equal to 1 ($h = 1$). With that antenna it can be obtain from Figure 3 the reflection level of the global system for each filter order. Figure 6 shows this matching level with respect to the filter order. In blue, there is the level with a Tchebyshev filter,

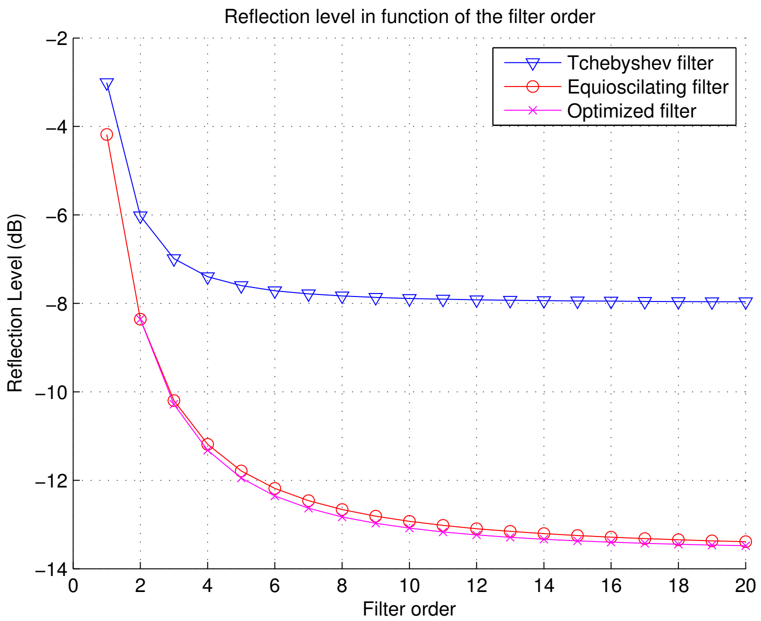


Figure 6: Reflection level of the global system G_{11} with a load of derivative $h = 1$.

in red the level with a filter that equioscillates and the magenta lines represent the reflection level obtained by solving 70 starting from the equioscillating filter. We can see that the best level is around -8dB for a Tchebyshev filter and around -13.5dB for an equioscillating filter. Figure 7 compares the response obtained with a Tchebyshev response and the equioscillating one, as well as the result obtained after minimizing the pseudohyperbolic distance with filters of different degree and a derivative of the load $h = 1$.

It is important to be noted that the Tchebyshev filter is not optimum even when the filter order tends to infinity meanwhile the level obtained with an equioscillating filter approaches to the level found by optimization when the order increases. In addition, the maximum different between the last two is about 0.2dB, what means that the level obtained by (63) is a good approximation to the minimum obtained by (70) even for filters of low order.

4 Considerations about the load

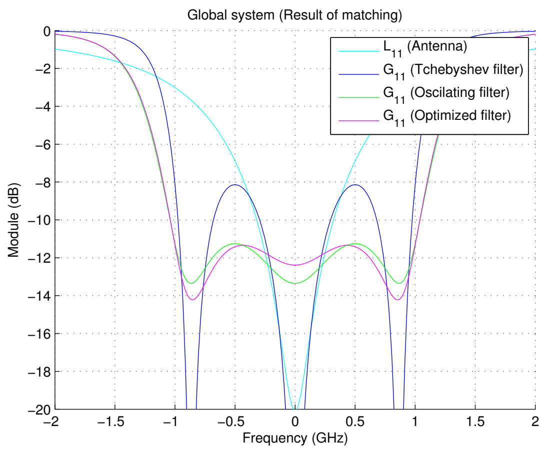
According to the previous results, the goal now is to characterize a load that provides the higher derivative possible. We will assume that at infinity $L_{11} = -1$ and $L_{22} = 1$ what corresponds to the notation in Figure 1 (polynomials ψ and φ are assumed to be monic). The polynomial are:

$$\varphi(\omega) = \omega - a = \varphi_1\omega + \varphi_0 \tag{72}$$

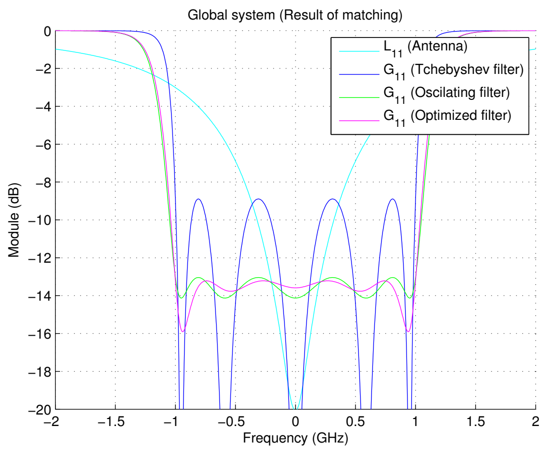
$$\rho(\omega) = \rho \tag{73}$$

$$\psi\psi^*(\omega) = \varphi\varphi^*(\omega) + \rho^2 \tag{74}$$

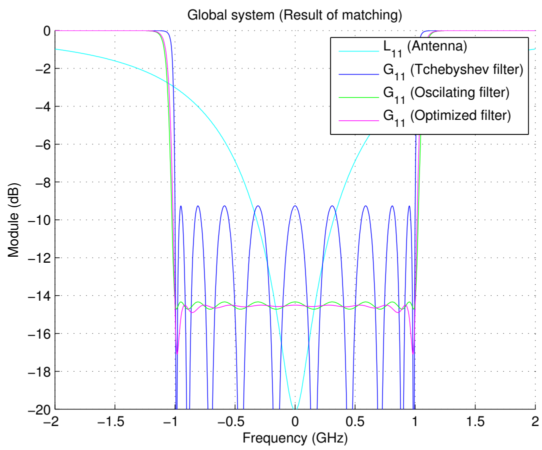
where $a = a_R + ja_I$ is the position of the reflection zero (a_R is the position along the frequency axis ω and a_I corresponds to the imaginary part) and ρ is related to the bandwidth. For the case of order 1, the



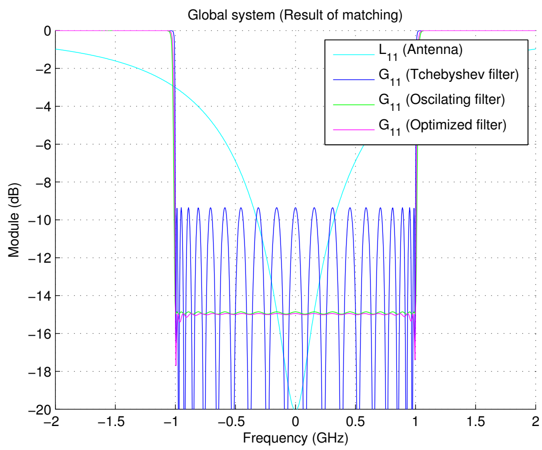
(a) Global system of degree $N = 3$.



(b) Global system of degree $N = 5$.



(c) Global system of degree $N = 10$.



(d) Global system of degree $N = 20$.

Figure 7: Comparison between a Tchebyshev filter, an equioscillating filter and the result obtained after optimizing the pseudohyperbolic distance.

polynomial ψ can be easily computed. Lets define the polynomial $\Psi(\omega) = \psi\psi^* = \Psi_2\omega^2 + \Psi_1\omega + \Psi_0$. Then:

$$\Psi_2 = \psi_1\psi_1^* = 1 \tag{75}$$
$$\Psi_1 = 2 \cdot Re\,(\psi_1\psi_0^*) = 2 \cdot Re\,(\varphi_1\varphi_0^*) \longrightarrow Re\,(\psi_0) = Re\,(\varphi_0) \tag{76}$$
$$\Psi_0 = |\psi_0|^2 = |\varphi_0|^2 + \rho^2 = a_R^2 + a_I^2 + \rho^2 \longrightarrow \mathrm{Im}(\psi_0) = \pm\sqrt{k^2 a_I^2 + 1} \tag{77}$$

Since the roots of ψ must lie in the upper half plane (positive) we have to chose the negative sign for $Im(\psi_0)$. Thus:

$$\psi = \omega - \left(a_R + j\sqrt{a_I^2 + \rho^2}\right) \tag{78}$$

Thus the derivative of L_{22} at infinity remains at follows:

$$\left.\frac{\partial L_{22}}{\partial \omega}\left(\frac{1}{\omega}\right)\right|_{\omega=0} = \varphi_0 - \psi_0 = -a_R - ja_I + a_R + j\sqrt{a_I^2 + \rho^2} = j\left(\sqrt{a_I^2 + \rho^2} - a_I\right) = jh \tag{79}$$

showing that the derivative at infinity is effectively pure imaginary. The imaginary part is:

$$h = \sqrt{a_I^2 + \rho^2} - a_I \tag{80}$$

This result for the derivative does not depend on the real part of the reflection zero of the load. It means that the theoretical matching level will be the same regardless of the position of the resonance of the antenna along the frequency axis. However, we can see that the derivative is bigger when ρ increases. This fact was obvious since when ρ goes to infinity, the reflection level goes to zero. It means that the antenna is already match and then the derivative tends to infinity. With regard to the value of a_I, if it is negative the derivative will increase when a_I increases. Thus the reflection zero of L_{22} should be in the lower half-plane, or equivalently the zero of L_{11} must be in the upper half plane. This is, the half-plane where the roots of ψ lie.

We attend now to the impedance of the antenna Z_{in}, We consider the input impedance of first degree with a pole in the imaginary axis at $\omega = Q = Q_R + jQ_I$ (K real negative and Q_I real positive):

$$Z_{in} = \frac{jK}{\omega - Q} \tag{81}$$

The reflection parameter in the input port can be directly computed as:

$$L_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{jK - (\omega - jQ)Z_0}{jK + (\omega - jQ)Z_0} = \frac{-\varphi*}{\psi} \tag{82}$$

Then it is possible to obtain the parameter $L_{22} = \varphi/\psi$ and its derivative as the difference between the second coefficient of each polynomial ($jh = \varphi_0 - \psi_0$):

$$L_{22} = \frac{jK + (\omega - Q_R + jQ_I)Z_0}{jK + (\omega - Q_R - jQ_I)Z_0} = \frac{\omega + j\left(\frac{K}{Z_0} - Q_R + jQ_I\right)}{\omega + j\left(\frac{K}{Z_0} - Q_R - jQ_I\right)} = \frac{\omega + \varphi_0}{\omega + \psi_0} \tag{83}$$
$$\left.\frac{\partial L_{22}\left(\frac{1}{\omega}\right)}{\partial \omega}\right|_{\omega=0} = \varphi_0 - \psi_0 = 2Q_I j = jh \tag{84}$$

Consequently, the value of h does only depend on the imaginary part of the pole of the input impedance (Q_I). In order to obtain a high derivative of L_{22} at infinity, the impedance Z_{in} must have a pole Q far away from the real axis. We can also consider the derivative of the phase of Z_{in} at $\omega = Q_R$ (the projection of the pole to the frequency axis). The derivative of the phase can be computed as the imaginary part of:

$$\frac{\partial}{\partial \omega} log\ Z_{in} = \frac{-1}{(\omega - Q)^2} = -\frac{\omega - Q_R + jQ_I}{(\omega - Q_R)^2 + Q_I^2} \tag{85}$$

Thus the imaginary part at the frequency $\omega = Q_R$:

$$\mathrm{Im}\frac{\partial}{\partial \omega} log\ Z_{in} = \frac{-1}{Q_I} = \frac{-2}{h} \tag{86}$$

This is equivalent to say that when h increases, the derivative of the phase goes to zero if we just regard the input impedance of the antenna at real frequencies. Using this result, we can express the best matching level that can be obtained by a given load in term of the derivative of the phase of Z_{in} at the frequency $\omega = Q_R$.

Additionally, if the quality factor of the antenna ($\mathcal{Q}_{\mathcal{F}}$) is defined as [Yaghjian and Best, 2005]:

$$\mathcal{Q}_{\mathcal{F}} \approx \frac{\omega_0}{2R(\omega_0)} \left|Z'_{in}(\omega_0)\right| \tag{87}$$

where $Z_{in}(\omega) = R(\omega) + jX(\omega)$. Then by defining ω_0 as the tuning frequency of the antenna where the input reactance $X(\omega_0)$ cancels ($Z_{in}(\omega_0) = R(\omega_0) > 0$):

$$\mathcal{Q}_{\mathcal{F}} \approx \frac{\omega_0}{2} \left|\frac{Z'_{in}(\omega_0)}{Z_{in}(\omega_0)}\right| = \frac{\omega_0}{2} \left|\frac{\partial}{\partial \omega} log\ Z_{in}(\omega_0)\right| \tag{88}$$

and inserting (85):

$$\mathcal{Q}_{\mathcal{F}} \approx \frac{\omega_0}{2} \frac{1}{\sqrt{(\omega_0 - Q_R)^2 + Q_I^2}} = \frac{\omega_0}{2} \frac{1}{\sqrt{(\omega_0 - Q_R)^2 + Q_I^2}} = \frac{\omega_0}{\sqrt{4(\omega_0 - Q_R)^2 + h^2}} \tag{89}$$

This expression shows that when h goes to infinity, the quality factor will decrease and vice versa. In addition, in the particular case when the antenna is tuned at the projection of the pole to the axis ($Q_R = \omega_0$), h is just expressed as the inverse of the quality factor times the frequency ω_0.

$$\mathcal{Q}_{\mathcal{F}}\left(Q_R = \omega_0\right) \approx \frac{\omega_0}{h} \tag{90}$$

As an example, Figure 8 shows the input reflection and the input impedance of two loads, the first one with a reflection zero in the upper half plane ($a = 0.5j$) and the second one in the lower half ($a = -0.5j$).

The first one provides a derivative $h = 1.22$ while for the second one $h = 0.82$. It can be noticed that although both have the same modulus, when the reflection zero of L_{11} lies in the lower half plane *(red lines)* the phase is always decreasing meanwhile if the zero is in the upper semi-plane *(blue lines)* the phase increases locally.

With respect to the input impedance, its phase is flatter when considering a reflection zero of L_{11} in the upper semi-plane. It means that the effect of the pole of Z_{in} (Q) is smaller. In the limiting case, when the pole of the impedance goes to $j\infty$, the antenna will show a constant impedance. It means that the antenna is already matched to this impedance (even if it is complex) and the matching filter will become just a regular filter matched to the same impedance.

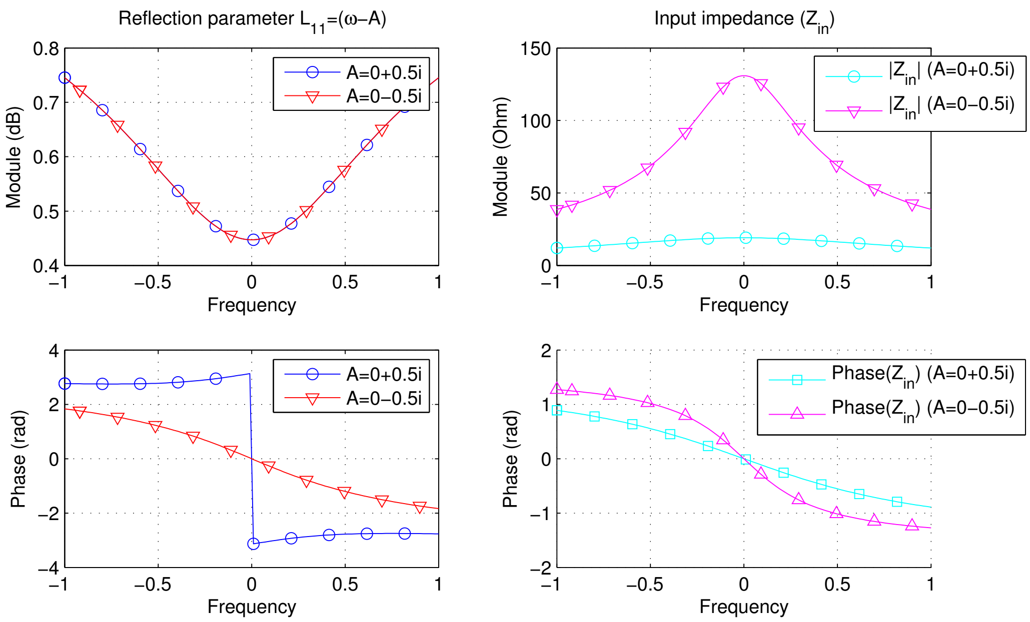


Figure 8: Reflection parameter (L_{11}) and input impedance (Z_{in}) for a load with the reflection zero in the upper half plane ($A = 0.5j$) and in the lower half plane ($A = -0.5j$).

4.1 Considerations about the filter sensibility

It has been shown that in theory the achievable matching level does not depend on the matching level showed by the antenna (level of L_{11}). In fact the matching level is not affected by the position along the real axis of the resonance provided by the antenna. Moreover the matching level will be improved if the reflection zero of the antenna goes into the complex plane toward $\omega = j\infty$, what means that the reflection of the antenna tends to 1.

However, in addition to the previous remark, it should be noted that when the level of the L_{11} tends to modulus 1, the admissible error in the matching filter tends to 0. In this case the tolerance of the filter components or even the error in the measurements will make the matching not feasible.

In Figure 9 we can see the allowed regions for the S_{22} that make the hyperbolic distance $D(S_{22}, L_{11}^*) < \delta$ (equation (9)) for different values of L_{11}. In other words, in order to obtain a given matching level below δ_1, it will be necessary for the S_{22} to be inside the first blue circle. It can be seen that when the modulus of the antenna reflection approaches to 1, the allowed error for S_{22} (region inside the first blue circle) decreases. Thus if we consider the practical implementation, it will be desirable to obtain at the same time the small reflection level possible within the passband.

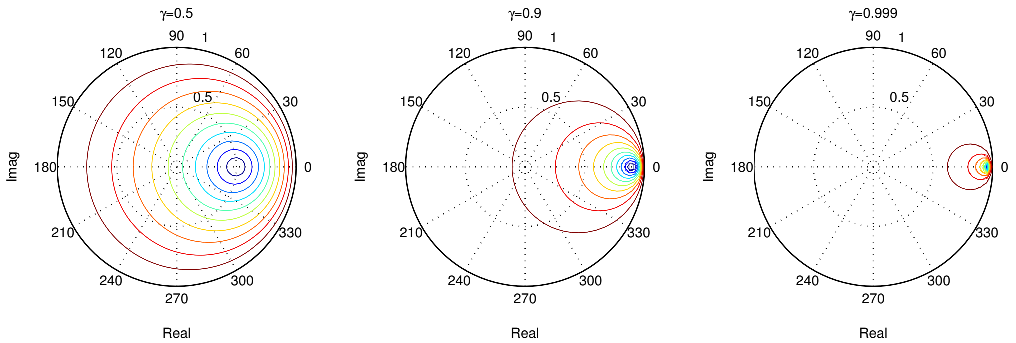


Figure 9: Equidistant lines to a point γ in the pseudohyperbolic metric.

5 Application for a dual-band antenna

Given the reflection parameter L_{11} of an antenna (Figure 10), even if it is dual-band, it is possible to apply both Problem 2 and Problem 1 to obtain the filter that achieves the best matching in both bands.

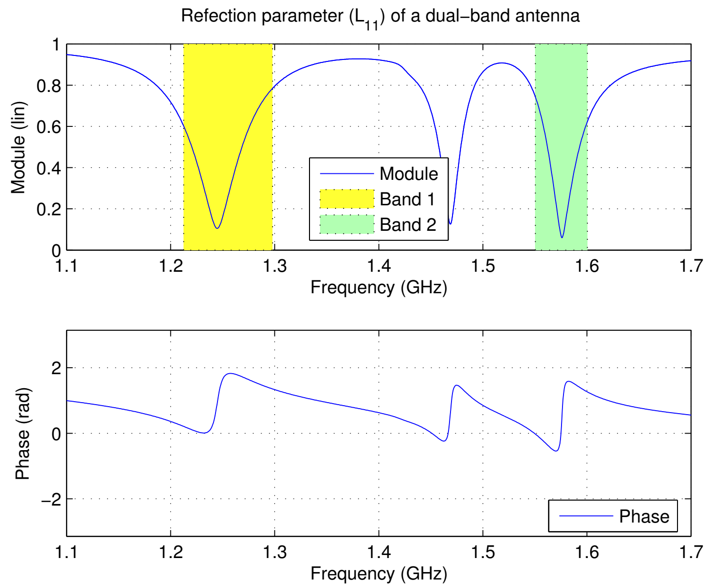
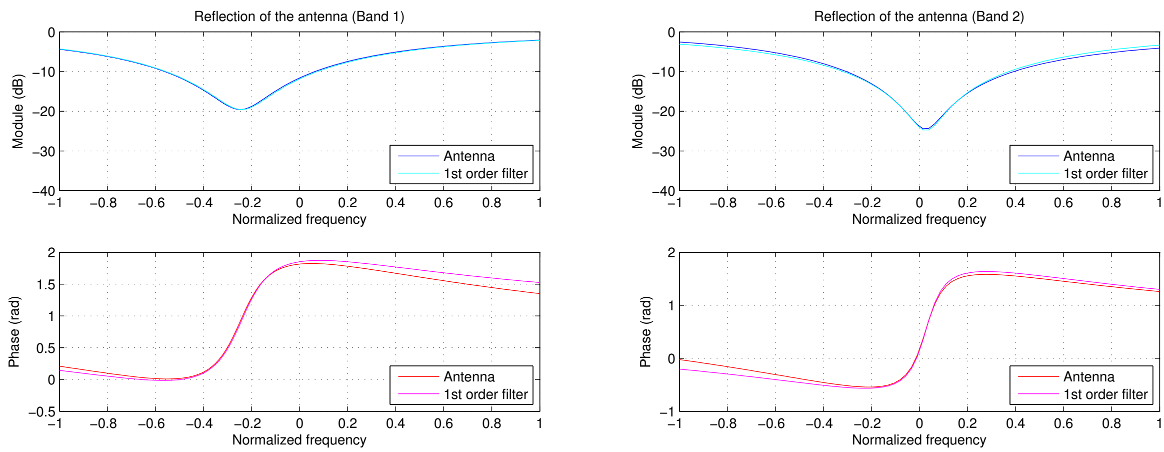


Figure 10: Reflection coefficient L_{11} of a dual-band antenna.

First, each band is normalized between $\omega = -1$ and $\omega = 1$. Since the band are considerable distant from each other the first attempt is to consider them separately and to do the rational approximation with a filter of first order for each one. Figures 11a and 11b show the reflection of the antenna normalized in both bands as well as the first order approximation in each one. In both cases the derivative of the obtained one-order load is around $h = 1.11$ what means that the best reflection level that can be obtained with an equioscillating filter of infinity degree is approximately $-15dB$ (Figure 3). Figure 12 shows the oscillating filter obtained in each band when considering only the single-band problem.



(a) Band 1 (Derivative $h = 1.1109$).

(b) Band 2 (Derivative $h = 1.1143$).

Figure 11: Approximation in each band of the dual-band antenna with a first order filter.

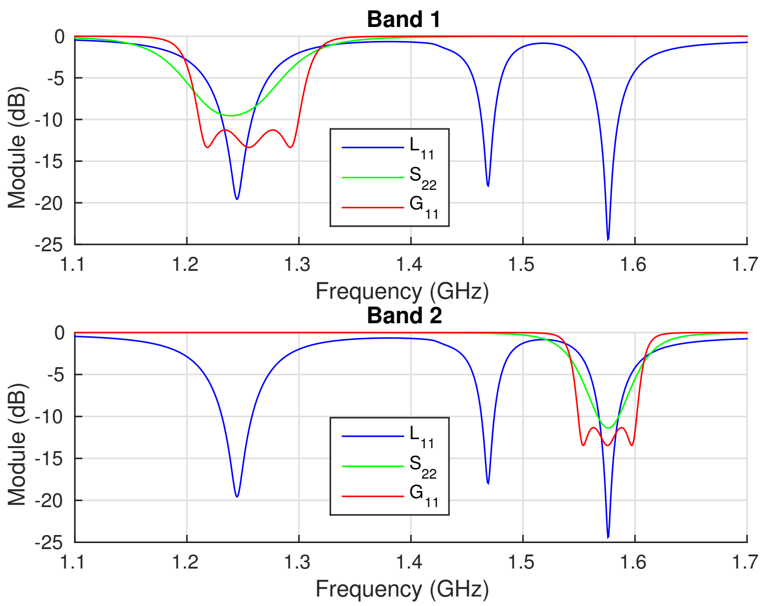


Figure 12: Reflection level for a single-band filter.

Following it is intended to obtain a dual-band filter. However, if we want to stick to the inline topology, we will face several problems. Lets consider that we have two inline single band filters (S^1 and S^2) with the same value of S_{11} at infinity ($S^1_{11}(\infty) = S^2_{11}(\infty)$) and the same transmission constant $r_1 = r_2$:

$$S^1 = \frac{1}{q_1}\begin{pmatrix} -p_1^* & r_1 \\ r_1 & p_1 \end{pmatrix} \qquad S^2 = \frac{1}{q_2}\begin{pmatrix} -p_2^* & r_2 \\ r_2 & p_2 \end{pmatrix} \tag{91}$$

then we can obtain a dual-band filter S by combining the roots of p_1 and p_2:

$$S = \frac{1}{q}\begin{pmatrix} -p^* & r \\ r & p \end{pmatrix} \tag{92}$$

where the roots of p are both the roots of p_1 along with the roots of p_2 and r is adjusted to approach the reflection level of the original single-band filters. This adjustment can only be performed if $r_1 = r_2$. Otherwise, with only one parameter (r) in the dual-band filter, it would not be possible to adjust the reflection level in each band independently.

Thus in order to obtain the dual-band filter it is necessary that the single-band filters corresponding to each band have both the same transmission constant ($r_1 = r_2$) what will not happen normally. In addition the reflection coefficient of each matching filter will not be 1 at infinity but they will show a phase. Therefore it is not possible to build a dual-band filter by combining them while keeping the phase at infinity of each single-band filter.

5.1 Rational approximation of the dual-band antenna

In order to solve the minimization of the hyperbolic distance (70) also for the dual-band antenna, it is necessary a rational approximation that expresses the load in term of the polynomials φ, ψ and ρ.

The frequency bands where the approximation is performed correspond to:

Band	f_{min} (GHz)	f_{max} (GHz)
L2	1.212	1.242
E6	1.258	1.298
L1	1.555	1.592

Next we show the obtained result for the approximation when considering a load L of different degrees.

5.1.1 Approximation of degree 2

We do the approximation of the antenna with a filter of degree 2, with 2 reflection zeros in the complex plane and a transmission zero in the real axis at $\omega = 1.4441$. The obtained polynomials are:

$$\varphi = (0.7286 + 0.6849j)z^2 - (2.0499 + 1.9401j)z + 1.4217 + 1.3559j$$
$$\rho = 0.0675z - 0.0975$$
$$\psi = z^2 - (2.8224 + 0.0682j)z + 1.9646 + 0.0986j$$

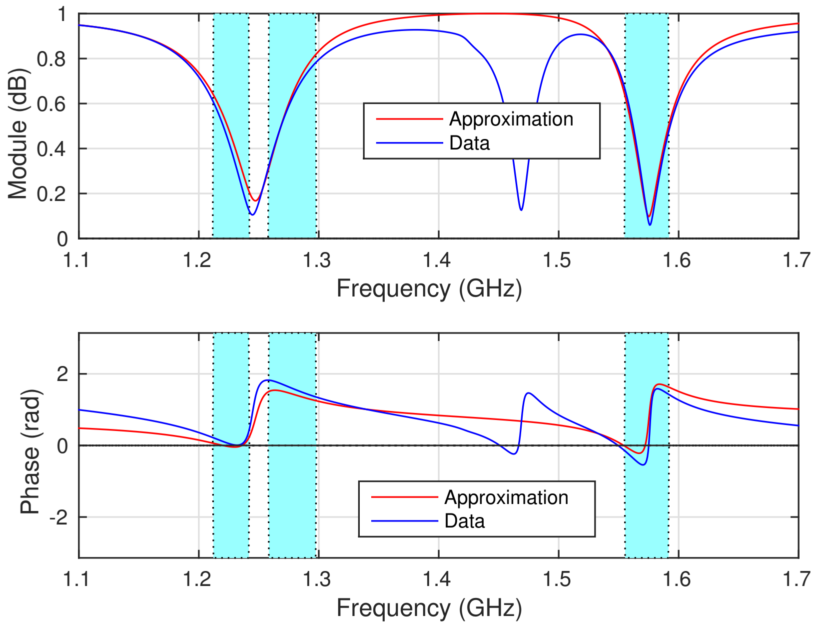


Figure 13: Load approximation with an order 2 reciprocal system.

5.1.2 Approximation of degree 3

It can be noticed that the result with the previous approximation is not quite good. It means that the load is not actually of degree 2. Therefore another approximation is performed, this time with a filter of degree 3 to obtain a load with 3 reflection zeros and no transmission zeros at finite frequencies. This time the approximation is better, then it is a good option in order to solve the approximation problem

$$\varphi = (0.9320 - 0.3626j)z^3 - (3.3827 - 1.5917j)z^2 + (3.9544 - 2.3190j)z - 1.4787 + 1.1201j$$
$$\rho = 0.0061$$
$$\psi = z^3 - (3.7296 + 0.3296j)z^2 + (4.5049 + 0.8990j)z - 1.7543 - 0.6032j$$

As a result, for the proposed antenna, it is necessary a load L of degree 3 to obtain a good approximation.

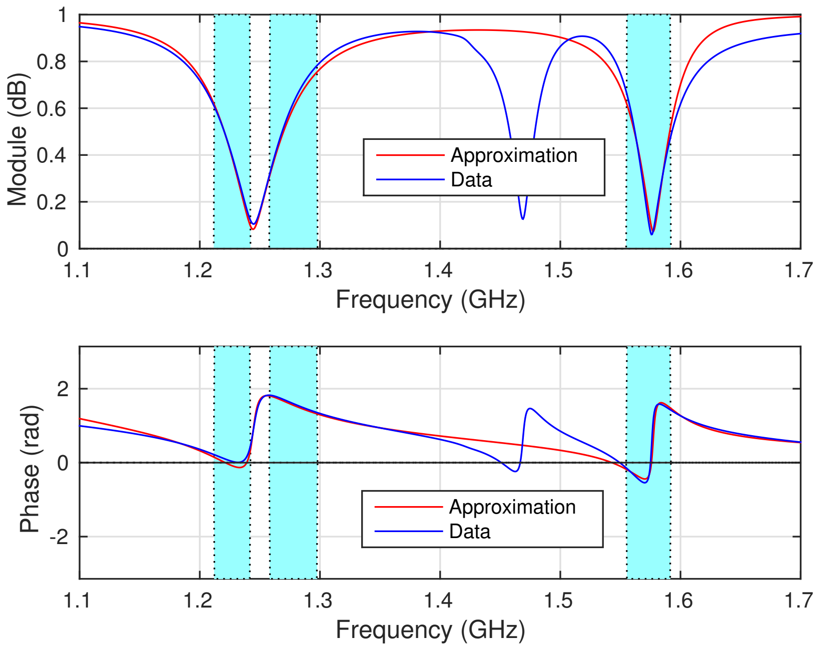


Figure 14: Load interpolation with an order 3 reciprocal system.

5.2 Dual-band filter

For a dual-band filter, we can still consider the the point-wise solution of problem 1. In this way it is possible to obtain M+1 (being M the degree of the matching filter) adjustable matching points[1] in total between both passbands (result shown in Figure 15 for a filter of degree $M = 4$). However the desired result of combining the matching filter for both bands (figure 12). would have M+2 adjustable matching points ($\frac{M}{2} + 1$ in each band being $\frac{M}{2}$ the degree of the matching filter for each single band.). In fact the global system G will be at least of degree $M + 2$ for a load of order 2. We can see in Figure 15 one extra reflection zero of G_{11} around 1.65 GHz, however due to the dimension of the interpolation problem, only $M + 1$ zeros of G_{11} will be adjustable.

It will only be possible to adjust $M + 2$ matching points when both passbands are symmetrical. Then the matching filter will be the same for both bands and it will be possible to obtain the dual-band matching filter by applying a quadratic transformation. In any case, we can use the result in (Figure 15) as initial point to the problem of minimizing the pseudo-hyperbolic distance. It it the same approach as the one performed in the single-band case by using the equioscillating filter as initial point. The obtained response is shown in Figure 16. By solving the minmax problem, we obtain a reflection level below 8 dB that is quite good even if it does not reach the result obtained when considering only one single band.

[1]Note that in this section we speak of "adjustable matching points". This is because with a filter (F) of degree M and a load (L) of degree K, there will be always N=M+K matching points (the reflection zeros of the global system G). However with a filter of degree M it is only possible to control the position of M+1 points (what agrees with theorem 1).

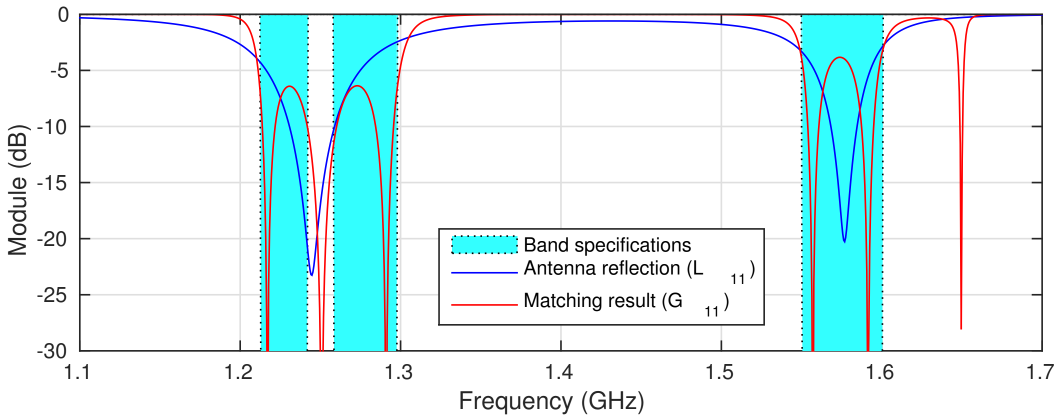


Figure 15: Point-wise solution of the little problem for the dual-band antenna.

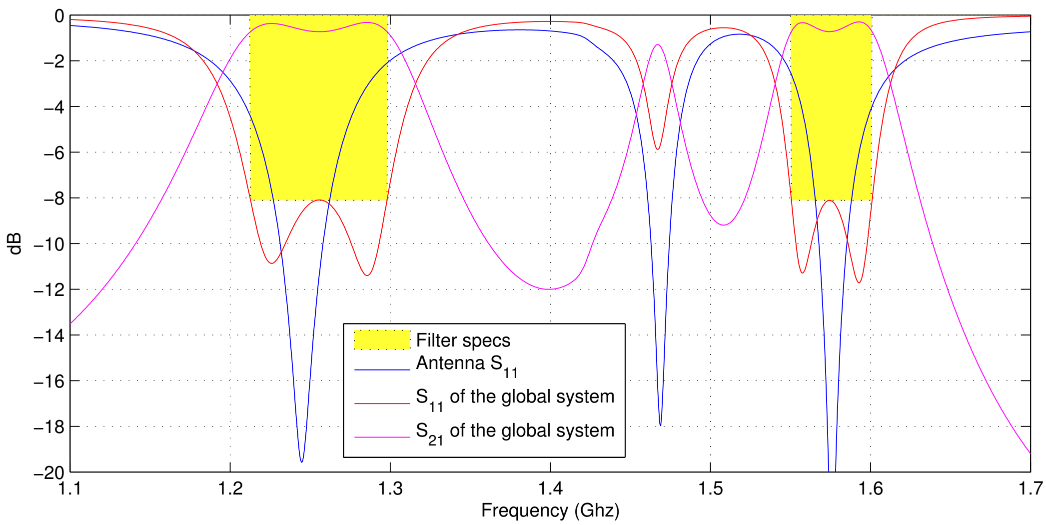


Figure 16: Result of the minmax optimization with the dual-band antenna.

5.3 Use of the transmission zeros for the interpolation problem

As it is already commented, it is only possible to control N matching point (complex conditions) when there are N complex parameters to adjust. For the interpolation problem, we had N matching point with a polynomial of degree $N-1$ by adjusting N complex parameters, namely the N coefficients of p.

Now it is intended to further add an extra adjustable matching points in the interpolation problem, then it will be necessary another complex parameter. If we now add two finite transmission zeros to the global system G, they will provide two extra real parameter needed to obtain a second extra point. Then the previous procedure is repeated. We do a point-wise matching for the dual-band antenna and then solve the problem 1 of minimizing the pseudohyperbolic distance (considering the rational approximation of order 3 for the antenna). Now the result is the one shown in Figure 17 that agrees better with the matching level obtained in the previous section for each band when considering the single-band problem.

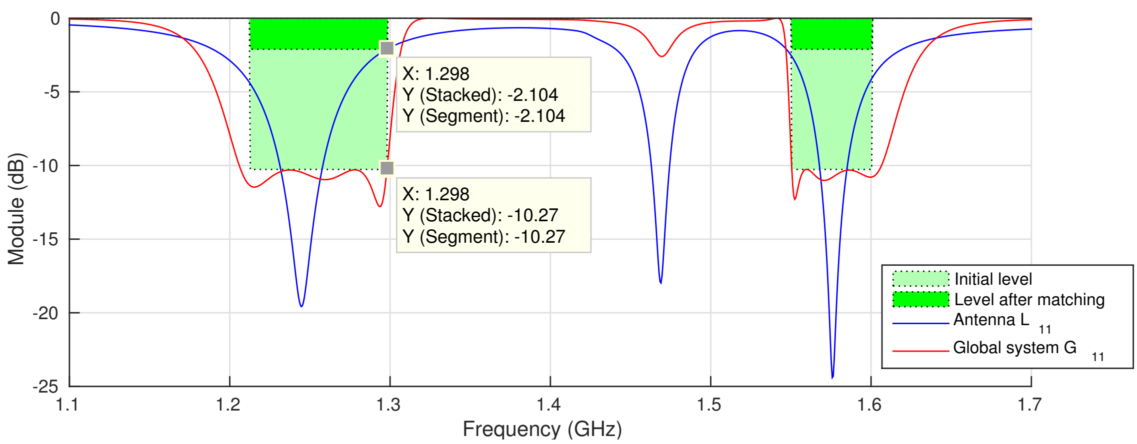


Figure 17: Improved band-reflection with the matching filter.

It is obtained a reflection level of $-10.27\ dB$ at the band edge that represent only the 9.4% of the power. This result is 6.5 times smaller than the original reflection value without matching (-2.104 dB) and provides a quite good matching even if the losses for the matching filter are considered.

5.4 Matching filter

The matching filter for the given antenna is obtained as a result of the above problem. With the usual notation for the filter F, we obtain the following values for the coefficients of p, q and r.

$$p(\omega) = \boldsymbol{e^{2.8607j}}\,(\omega^4 + (-5.6267 + 0.0433i)\omega^3 + (11.8299 - 0.1885i)\omega^2 + (-11.0143 + 0.2739i)\omega + 3.8321 - 0.1326i) \tag{93}$$

$$q(\omega) = \omega^4 + (-5.6267 - 0.2652i)\omega^3 + (11.7956 + 1.1304i)\omega^2 + (-10.9152 - 1.6009i)\omega + 3.7607 + 0.7532i \tag{94}$$

$$r(\omega) = 0.0416\omega^2 + -0.1193\omega + 0.0851 \tag{95}$$

The scattering parameters of the matching filter are shown in Figure 18.

As the last remark, it should be noticed the constant phase shifting in front of the polynomial p. It means that the coefficient $S_{22} = \frac{p}{q}$ will not take the value 1 at infinity but the previous one ($e^{2.8607j}$). For the practical point of view, a phase shifting at a given frequency can be implemented with a transmission line of the corresponding length. However in this case, given the width of the frequency range that is considered (from 1.212 to 1.590 Ghz), a generic phase shifting element will be added. Thus the system remains:

In Figure 19, the filter represents now the previous matching filter where the constant phase at infinity has been taken out. In this way it is possible to represent the filter by its coupling matrix:

$$M = \begin{bmatrix} 0 & -0.6648 & 0.5512 & -0.1024 & 0.1978 & 0 \\ -0.6648 & 1.1741 & 0 & 0 & 0 & 0.4524 \\ 0.5512 & 0 & -0.9988 & 0 & 0 & 0.4568 \\ -0.1024 & 0 & 0 & -0.7165 & 0 & 0.1942 \\ 0.1978 & 0 & 0 & 0 & 0.5376 & 0.3481 \\ 0 & 0.4524 & 0.4568 & 0.1942 & 0.3481 & 0 \end{bmatrix} \tag{96}$$

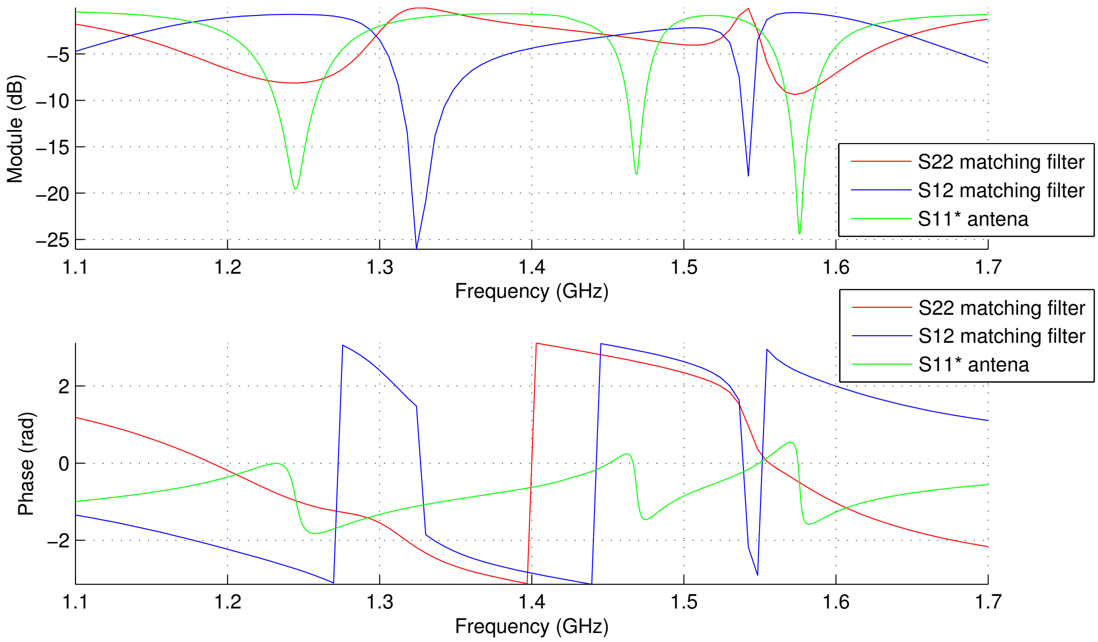


Figure 18: Matching filter.

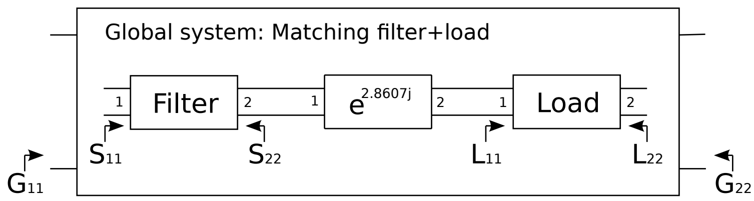


Figure 19: Global system with a phase shifter element.

6 Conclusions and perspectives

A synthesis procedure for microwave filtering functions matched to a frequency varying load has been reviewed. We have obtained closed formulas to approximate the bound of the reflection level for the single-band case with a load of degree 1 as well as an analytic expression for the scattering parameters in this cases. It has been showed, for the single-band case, that the best matching does not depends on the position of the resonance of the antenna along the frequency axis. In the same way, we show that the best antenna in term of matching is not the one that is perfectly match in the frequency but the one with an smaller quality factor. Thus it will be necessary to reach a trade-off between the quality factor of the antenna and the achievable level of matching.

In addition, we have developed and algorithm of rational approximation that minimizes the pseudohyperbolic distance to a given function. With that algorithm we are able to obtain the optimum matching filter in order to verify the accuracy of the previous formulas as well as to apply the synthesis approach to the case of a multiband antenna of degree 3. In all cases we show by using the proposed matching filter, a considerable improvement of the reflection level at the edge of the band. In particular, for the dual band antenna, we obtain a reflection level of approx. -10 dB, more than 6 times better that the original reflection level of the antenna (approx. -2.1 dB).

We have also showed that in particular cases, it is also possible to use the transmission zeros of the matching filter to adjust a larger number of matching points. In this case, it is possible to obtain N points of perfect matching to a given load with a filter of degree less than $N-1$. It means that we also use the resonance of the antenna to build a filter of higher degree than the actual matching filter. Nevertheless it is still needed a further research in this field in order to identify the requirement for the load that allow to control a given number of extra matching points.

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